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HORN'S PROBLEM, AND FOURIER ANALYSIS

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Abstract Let A and B be two $n \times n$ Hermitian matrices. Assume that the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A are known, as well as the eigenvalues $\beta_1, \ldots, \beta_n$ of B. What can be said about the eigenvalues of the sum $C = A + B$? This is Horn's problem. We revisit this question from a probabilistic viewpoint. The set of Hermitian matrices with spectrum $\{\alpha_1, \ldots, \alpha_n\}$ is an orbit $\mathcal{O}_\alpha$ for the natural action of the unitary group $U(n)$ on the space of $n \times n$ Hermitian matrices. Assume that the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$, and, independently, the random Hermitian matrix Y is uniformly distributed on $\mathcal{O}_\beta$. We establish a formula for the joint distribution of the eigenvalues of the sum $Z = X + Y$. The proof involves orbital measures with their Fourier transforms, and Heckman's measures.

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Introduction

Consider two Hermitian matrices A, B, and their sum $C = A + B$. Assume that the eigenvalues $\alpha_1, \ldots, \alpha_n$ of A are known, and the eigenvalues $\beta_1, \ldots, \beta_n$ as well. Here is Horn's problem : what can be said about the eigenvalues $\gamma_1, \ldots, \gamma_n$ of C ? Horn's conjecture [1962] says that the set of possible eigenvalues $\gamma_1, \ldots, \gamma_n$ for C is determined by a family of inequalities of the form

$$\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j,$$

for certain "admissible" triples (I, J, K) of subsets of $\{1, 2, \ldots, n\}$. Weyl inequalities are of this type [Weyl,1912]. Klyachko describes these admissible triplets in terms of Schubert calculus [1998]. To a subset $I \subset \{1, \ldots, n\}$ one associates a Schubert variety. The admissible triplets are those for which the associated Schubert varietties have a non empty intersection. We will not go further in this direction. See for instance the survey paper [Bhatia,2001].

It is possible to consider Horn's problem from a probabilistic point of view (See [Frumkin-Goldberger,2006], and [Zuber,2017]). The set of $n \times n$ Hermitian matrices X with eigenvalues $\alpha_1, \ldots, \alpha_n$ is an orbit $\mathcal{O}_\alpha$ for the action of the unitary group $U(n)$. Assume the random Hermitian matrix X to be uniformly distributed on $\mathcal{O}_\alpha$, and, independently, the matrix Y uniformly distributed on $\mathcal{O}_\beta$. The question is now : what is the distribution of the eigenvalues $\gamma_1, \ldots, \gamma_n$ of the sum $Z = X + Y$? We follow this approach and in this paper we determine explicitely this distribution $\nu_{\alpha,\beta}$.

The proof uses the celebrated Harish-Chandra-Itzykson-Zuber integral, and Heckman's measures. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, the orbit

$$\mathcal{O}_\alpha = \{U \mathrm{diag}(\alpha_1, \ldots, \alpha_n) U^* \mid U \in U(n)\},$$

carries a natural probability, the orbital measure μ_α. The Fourier-Laplace transform of μ_α is given by the Harish-Chandra-Itzykson-Zuber formula. Heckman's measure M_α is the projection of the orbital measure μ_α on the space of diagonal matrices. Heckman studied this measure in a more general setting, and gave an explicit formula for it [1982]. Our main result is an explicit formula for the distribution $\nu_{\alpha,\beta}$ (Theorem 4.1) :

$$\nu_{\alpha,\beta} = C_n V_n(x) \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \delta_{\sigma(\alpha)} * M_\beta,$$

where V_n denotes the Vandermonde polynomial in n variables,

$$V_n(x) = \prod_{i<j} (x_i - x_j),$$

and $\mathfrak{S}_n$ is the symmetric group which acts on $\mathbb{R}^n$ as follows :

$$\sigma\big((x_1, \ldots, x_n)\big) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The support $S(\alpha, \beta)$ of the measure $\nu_{\alpha,\beta}$ is the set of possible systems of eigenvalues for the matrix $C = A + B$, if $\alpha_1, \ldots, \alpha_n$ are the eigenvalues of A, and $\beta_1, \ldots, \beta_n$ the eigenvalues of B.

Horn's problem is related to representation theory. If α and β are highest weights of two irreducible representations π_α and π_β of the unitary group $U(n)$, the spectrum of the tensor product $\pi_\alpha \otimes \pi_\beta$ is contained in the support of $\nu_{\alpha,\beta}$. But we will not consider this aspect or Horn's problem. See [Fulton,1998], [Fulton,2000], [Knutson-Tao,1999], [Knutson-Tao-Woodward,2004].

We introduce in Section 1 the orbital measures on the space of Hermitian matrices, and the radial part of a measure which is invariant under the action of the unitary group. In section 2 we recall the Harish-Chandra-Itzykson-Zuber integral, and, in Section 3, some properties of Heckman's measures. We state and prove our main result in Section 4. The case of a rank one matrix B is considered in Section 5, and our result is compared to results of Frumkin and Goldberger. In last Section we give some formulas related to the case of 2×2 real symmetric matrices. We conclude with a few remarks.

1 Orbital measures

Let $\mathcal{H}_n(\mathbb{R}) = Sym(n, \mathbb{R})$, the space of $n \times n$ real symmetric matrices, and $\mathcal{H}_n(\mathbb{C}) = Herm(n, \mathbb{C})$, the space of $n \times n$ Hermitian matrices. For a matrix $X \in \mathcal{H}_n(\mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) the classical spectral theorem says that the eigenvalues are real and the corresponding eigenspaces are orthogonal. We will denote by D_n the space of real diagonal matrices, $D_n \simeq \mathbb{R}^n$, and define the chamber

$$C_n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_1 \geq t_2 \geq \cdots \geq t_n\}.$$

Let $U_n(\mathbb{F}) = O(n)$, the orthogonal group, and $U_n(\mathbb{C}) = U(n)$, the unitary group. The group $U_n(\mathbb{F})$ acts on the space $\mathcal{H}_n(\mathbb{F})$ by the transformations $X \mapsto UXU^*$ ($U \in U_n(\mathbb{F})$). Let $\mathcal{O}_\alpha$ denote the orbit of the diagonal matrix $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ with $(\alpha_1, \ldots, \alpha_n) \in C_n$:

$$\mathcal{O}_\alpha = \{UAU^* \mid U \in U_n(\mathbb{F})\}.$$

From the spectral theorem it follows that

$$\mathcal{O}_\alpha = \big\{X \in \mathcal{H}_n(\mathbb{F}) \mid \mathrm{spectrum}(X) = \{\alpha_1, \ldots, \alpha_n\}\big\}.$$

The orbit $\mathcal{O}_\alpha$ carries a natural probability measure : the orbital measure μ_α, image of the normalized Haar measure ω of the compact group $U_n(\mathbb{F})$ under the map

$$U_n(\mathbb{F}) \to \mathcal{H}_n(\mathbb{F}), \quad U \mapsto UAU^*.$$

For a continuous function f on $\mathcal{O}_\alpha$,

$$\int_{\mathcal{O}_\alpha} f(X)\mu_\alpha(dX) = \int_{U_n(\mathbb{F})} f(UAU^*)\omega(dU).$$

Let μ be a measure on $\mathcal{H}_n(\mathbb{F})$ which is invariant under $U_n(\mathbb{F})$. The integral of a function f can be decomposed as follows

$$\int_{\mathcal{H}_n(\mathbb{F})} f(X)\mu(dX) = \int_{\mathbb{R}^n} \Big(\int_{U_n(\mathbb{F})} f\big(U\,\mathrm{diag}(t_1,\ldots,t_n)\,U^*\big)\omega(dU)\Big)\nu(dt),$$

where ν is a measure on $\mathbb{R}^n$ which is invariant under the symmetric group $\mathfrak{S}_n$: for a function F on $\mathbb{R}^n$, and $\sigma \in \mathfrak{S}_n$,

$$\int_{\mathbb{R}^n} F(t_{\sigma(1)},\ldots,t_{\sigma(n)})\nu(dt) = \int_{\mathbb{R}^n} F(t_1,\ldots,t_n)\nu(dt).$$

The measure ν is called the *radial part* of the measure μ. If μ is a probability measure on $\mathcal{H}_n(\mathbb{F})$ which is $U_n(\mathbb{F})$-invariant, its radial part ν is the joint distribution of the eigenvalues of a random matrix X whose distribution is the measure μ. For instance, the radial part ν_α of the orbital measure μ_α is

$$\nu_\alpha = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \delta_{\sigma(\alpha)},$$

where $\sigma(\alpha) = \big(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(n)}\big)$. If the measure μ has a density h with respect to the Lebesgue measure m on the real vector space $\mathcal{H}_n(\mathbb{F}) : \mu(dX) = h(X)m(dX)$, then, by the Weyl integration formula,

$$\nu(dt) = C\,h(t)\,|V_n(t)|^d dt_1\ldots dt_n,$$

where V_n is the Vandermonde polynomial,

$$V_n(t) = \prod_{1\leq i<j\leq n}(t_i - t_j),$$

$d = 1$ if $\mathbb{F} : \mathbb{R}$, $d = 2$ if $\mathbb{F} = \mathbb{C}$, and C is a constant which depends on d and n. In this paper the radial part ν is defined as a $\mathfrak{S}_n$-invariant measure on $\mathbb{R}^n$. It is more usual to define the radial part as a measure on the chamber C_n. This is a slight difference, but responsible, in some explicit formulas, for the appearence of a factor $n!$ which does not show up in some other papers.

Assume that the random Hermitian matrix X is uniformly distributed on the orbit $\mathcal{O}_\alpha$, i.e. according to the orbital measure μ_α, and the random Hermitian matrix Y is uniformly distributed on $\mathcal{O}_\beta$, i.e. according to μ_β. Then the sum $Z = X+Y$ is distributed according to the convolution product $\mu_\alpha * \mu_\beta$ and the joint distribution of the eigenvalues of Z is equal to the radial part $\nu_{\alpha,\beta}$ of $\mu_\alpha * \mu_\beta$. In case of $\mathbb{F} = \mathbb{C}$ we will determine explicitely the measure $\nu_{\alpha,\beta}$ by using Fourier analysis (Theorem 4.1).

2 Fourier-Laplace transform

The Fourier-Laplace transform of a bounded measure μ on $\mathcal{H}_n(\mathbb{F})$ is given by

$$\mathcal{F}\mu(Z) = \int_{\mathcal{H}_n(\mathbb{F})} e^{\mathrm{tr}(ZX)} \mu(dX).$$

The function $\mathcal{F}\,\mu$ is defined on $i\mathcal{H}_n(\mathbb{F})$. If the support of μ is compact, then $\mathcal{F}\mu$ is defined on $Sym(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{R}$, on $M_n(n, \mathbb{C})$ if $\mathbb{F} = \mathbb{C}$. If the measure μ is $U_n(\mathbb{F})$-invariant, its Fourier-Laplace transform $\mathcal{F}\mu$ is $U_n(\mathbb{F})$-invariant as well, and determined by its restriction to the space D_n of diagonal matrices. For

$$\begin{aligned} Z &= \mathrm{diag}(z_1, \ldots, z_n), \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n, \\ T &= \mathrm{diag}(t_1, \ldots, t_n), \quad t = (t_1, \ldots, t_n) \in \mathbb{R}^n, \end{aligned}$$

define the function

$$\mathcal{E}_n(z, t) = \int_{U_n(\mathbb{F})} e^{\mathrm{tr}\,(ZUTU^*)} \omega(dU).$$

The Fourier-Laplace transform of a $U_n(\mathbb{F})$-invariant bounded measure μ can be written, for $Z = \mathrm{diag}(z_1, \ldots, z_n)$,

$$\mathcal{F}\mu(Z) = \int_{\mathbb{R}^n} \mathcal{E}_n(z, t) \nu(dt),$$

where ν is the radial part of μ. Observe that the Fourier-Laplace transform of the orbital measure μ_α is given by

$$\mathcal{F}\mu_\alpha(Z) = \mathcal{E}_n(z, \alpha).$$

Since $\mathcal{F}(\mu_\alpha * \mu_\beta) = \mathcal{F}\mu_\alpha\, \mathcal{F}\mu_\beta$, we obtain the following key relation for determining the measure $\nu_{\alpha,\beta}$.

Proposition 2.1. *The measure $\nu_{\alpha,\beta}$ is determined by the relation : for $z \in \mathbb{C}^n$,*

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

Observe that this relation is nothing but the product formula for the spherical functions of the following Gelfand pair (G,K) :

$$G = U_n(\mathbb{F}) \ltimes \mathcal{H}_n(\mathbb{F}), \quad K = U_n(\mathbb{F}).$$

The group G acts on $\mathcal{H}_n(\mathbb{F})$ by the transformations

$$g \cdot X = UXU^* + A \quad \big(g = (U,A)\big).$$

A function f on G which is K-biinvariant can be seen as a $U_n(\mathbb{F})$-invariant function on $\mathcal{H}_n(\mathbb{F})$, and such a function only depends on the eigenvalues. Hence we can identify a K-biinvariant function f on G to a $\mathfrak{S}_n$-invariant function F on $\mathbb{R}^n$:

$$f(g) = F(t_1,\ldots,t_n),$$

if $t_1,\ldots,t_n$ are the eigenvalues of $g\cdot 0$. The spherical functions of the Gelfand pair (G,K) are given by

$$\varphi_z(g) = \mathcal{E}(z,t) \quad \big(t = (t_1,\ldots,t_n),\ z \in \mathbb{C}^n\big).$$

They satisfy the functional equation :

$$\int_K \varphi_z(g_1 U g_2)\omega(dU) = \varphi_z(g_1)\varphi_z(g_2) \quad (g_1, g_2 \in G).$$

With the identification

$$\varphi_z(g_1) = \mathcal{E}(z,\alpha), \quad \varphi_z(g_2) = \mathcal{E}(z,\beta),$$

the functional equation can be written as

$$\int_{\mathbb{R}^n} \mathcal{E}_n(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}_n(z,\alpha)\mathcal{E}_n(z,\beta).$$

For this viewpoint see the inspiring paper [Berezin-Gelfand,1962]. See also the recent paper [Kuijlaars-Roman,2016]. Closely related is the paper [Graczyk-Sawyer,2002], and Section 7 in [Rösler,2003].

In case of $\mathbb{F} = \mathbb{C}$, there is an explicit formula for $\mathcal{E}_n(z,t)$, the Harish-Chandra-Itzykson-Zuber formula [Itzykson-Zuber,1980]. In fact it is a special case of a formula established by Harish-Chandra for the adjoint action of a compact Lie group on its Lie algebra [1957].

Theorem 2.2. *Let $A, B, \in \mathcal{H}_n(\mathbb{C})$ with eigenvalues $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$. Then*

$$\int_{U_n(\mathbb{C})} e^{\mathrm{tr}\,(AUBU^*)}\omega(dU) = \delta_n! \frac{1}{V_n(\alpha)V_n(\beta)} \det\bigl(e^{\alpha_i\beta_j}\bigr)_{1\leq i,j\leq n},$$

where $\delta_n = (n-1, n-2, \dots, 1, 0)$, $\delta_n! = (n-1)!(n-2)! \dots 2!$.

Then we get

$$\mathcal{E}_n(z,t) = \delta_n! \frac{1}{V_n(z)V_n(t)} \det\bigl(e^{z_i t_j}\bigr)_{1\leq i,j\leq n}.$$

The formula can also be seen as the Fourier-Laplace transform of an orbital measure :

$$\mathcal{F}\mu_\alpha(Z) = \delta_n! \frac{1}{V_n(z)V_n(\alpha)} \det\bigl(e^{z_i\alpha_j}\bigr)_{1\leq i,j\leq n},$$

for $Z = \mathrm{diag}(z_1, \dots, z_n)$.

3 Heckman's measure

Let us consider the projection q of the space $\mathcal{H}_n(\mathbb{F})$ onto the subspace $D_n \simeq \mathbb{R}^n$ of real diagonal matrices,

$$q : \mathcal{H}_n(\mathbb{F}) \to \mathbb{R}^n, \quad X \mapsto (x_1, \dots, x_n), \quad x_i = X_{ii}.$$

Recall the Horn's convexity theorem [Horn,1954] : the image $q(\mathcal{O}_\alpha)$ of the orbit $\mathcal{O}_\alpha$ is equal to the convex hull $C(\alpha)$ of the points $\sigma(\alpha)$,

$$q(\mathcal{O}_\alpha) = C(\alpha) := \mathrm{Conv}\bigl(\{\sigma(\alpha) \mid \sigma \in \mathfrak{S}_n\}\bigr).$$

From now on, in this section, we assume $\mathbb{F} = \mathbb{C}$. The image $M_\alpha = q(\mu_\alpha)$ of the orbital measure μ_α is called Heckman's measure. In fact this measure has been described by Heckman in a more general setting [1982] (see also [Duflo-Heckman-Vergne,1984]). The measure M_α has support $q(\mathcal{O}_\alpha)$ which is contained in the hyperplane $x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n$. It is symmetric, i.e. invariant under the group $\mathfrak{S}_n$, acting by permuting the coordinates. If the eigenvalues $\alpha_1, \dots \alpha_n$ are distinct, Heckman's measure M_α is absolutely continuous with respect to the Lebesgue measure of this hyperplane, and its density is piecewise polynomial. These facts have been established by

Heckman. Let us recall their proof in the present special case. For a bounded measure M on $\mathbb{R}^n$ we will denote by $\widehat{M}$ its Fourier-Laplace transform :

$$\widehat{M}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M(dx).$$

For $\alpha \in \mathbb{R}^n$ with the α_i all distinct, define the skew-symmetric measure

$$\eta_\alpha = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \delta_{\sigma(\alpha)}.$$

The Fourier-Laplace transform of η_α is given by

$$\widehat{\eta_\alpha}(z) = \frac{\delta_n!}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) e^{(z|\sigma(\alpha))} = \frac{\delta_n!}{V_n(\alpha)} \det\bigl(e^{z_i \alpha_j}\bigr)_{1 \le i,j \le n}.$$

The map $\alpha \mapsto \eta_\alpha$ extends as a continuous map $\mathbb{R}^n \to \mathcal{E}'(\mathbb{R}^n)$, the space of distributions on $\mathbb{R}^n$ with compact support. In particular

$$\eta_0 = V_n\Bigl(\frac{\partial}{\partial x}\Bigr)\delta_0.$$

Proposition 3.1. *Heckman's measure M_α satisfies the following equation*

$$V_n\Bigl(-\frac{\partial}{\partial x}\Bigr) M_\alpha = \eta_\alpha.$$

Proof. For a bounded measure μ on $\mathcal{H}_n(\mathbb{C})$, the Fourier-Laplace transform of the projection $M = q(\mu)$ of μ on D_n is equal to the restriction to D_n of the Fourier-Laplace transform of μ : $\widehat{M}(z) = \mathcal{F}\mu(Z)$, for $Z = \mathrm{diag}(z_1, \dots, z_n)$. Hence

$$\widehat{M_\alpha}(z) = \mathcal{F}\mu_\alpha(Z) = \mathcal{E}_n(z, \alpha).$$

Therefore, by the Harish-Chandra-Itzykson-Zuber formula (Theorem 2.2),

$$\widehat{M_\alpha}(z) = \delta_n! \frac{1}{V_n(\alpha)V_n(z)} \det\bigl(e^{z_i \alpha_j}\bigr)_{1 \le i,j \le n} = \frac{1}{V_n(z)} \widehat{\eta_\alpha}(z).$$

This equality, which can be written $V_n(z)\widehat{M_\alpha}(z) = \widehat{\eta_\alpha}(z)$, means an equality between two Fourier-Laplace transforms of compactly supported distributions, and implies the following differential equation

$$V_n\Bigl(-\frac{\partial}{\partial x}\Bigr) M_\alpha = \eta_\alpha. \qquad \square$$

For solving this equation we will use an elementary solution of the differential operator $V_n\big(\frac{\partial}{\partial x}\big)$. Let us define the distribution E_n on $\mathbb{R}^n$:

$$\langle E_n, \varphi\rangle = \int_{\mathbb{R}_+^{\frac{n(n-1)}{2}}} \varphi\Big(\sum_{i<j} t_{ij}\varepsilon_{ij}\Big) dt_{ij},$$

where $\varepsilon_{ij} = e_i - e_j$ ($\{e_1, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^n$).

Proposition 3.2. *The distribution E_n is an elementary solution of the differential operator $V_n\big(\frac{\partial}{\partial x}\big)$:*

$$V_n\Big(\frac{\partial}{\partial x}\Big) E_n = \delta_0.$$

The support of E_n is the convex cone in the hyperplane $x_1 + \cdots + x_n = 0$ generated by the vectors ε_{ij}, with $i < j$. The distribution E_n is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = 0$. The cone $\operatorname{supp}(E_n)$ *decomposes into a finite union of cones, and the restriction of the density to each of these cones is a polynomial, homogeneous of degree $\frac{1}{2}(n-1)(n-2)$.*

Proof. The differential opeartor $V_n\big(\frac{\partial}{\partial x}\big)$ is a product of degree one differential operators :

$$V_n\Big(\frac{\partial}{\partial x}\Big) = \prod_{i<j}\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big).$$

An elementary solution of $\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}$ is the Heaviside distribution Y_{ij} defined by

$$\langle Y_{ij}, \varphi\rangle = \int_0^\infty \varphi\big(t\varepsilon_{ij}\big) dt.$$

Hence the convolution product

$$E_n = \overset{*}{\prod_{i<j}} Y_{ij}$$

is an elementary solution of $V_n\big(\frac{\partial}{\partial x}\big)$. ∎

For a function φ define $\check{\varphi}(x) = \varphi(-x)$, and for a distribution T, $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$.

Theorem 3.3. *The Heckman measure M_α is given by*

$$M_\alpha = \check{E}_n * \eta_\alpha.$$

If the α_i are all distinct, the measure M_α is absolutely continuous with respect to the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n$, and the density is piecewise polynomial. This density is continuous for $n \geq 3$. The map $\alpha \mapsto M_\alpha$ extends as a continuous map $\mathbb{R}^n \to \mathcal{M}_c^1(\mathbb{R}^n)$, the set of probability measures on $\mathbb{R}^n$ with compact support.

Proof. Observe first the following fact : let F and G be distributions on $\mathbb{R}^n$. Assume the support of F to be compact. Let $D = P\big(\frac{\partial}{\partial x}\big)$ be a differential operator with constant coefficients. Then

$$DF * G = F * DG = D(F * G).$$

Therefore

$$\check{E}_n * V_n\Big(-\frac{\partial}{\partial x}\Big)M_\alpha = V_n\Big(-\frac{\partial}{\partial x}\Big)\check{E}_n * M_\alpha = M_\alpha.$$

By Proposition 3.1,

$$V_n\Big(-\frac{\partial}{\partial x}\Big)M_\alpha = \eta_\alpha.$$

Hence

$$M_\alpha = \check{E}_n * \eta_\alpha.$$ ∎

Example 1, n=2

The elementary solution E_2 is given by

$$\langle E_2, \varphi\rangle = \int_0^\infty \varphi\big(t\varepsilon_{1,2}\big)dt.$$

In the present case

$$\mathfrak{S}_2 = \{Id, \tau\},\ \tau : (x_1, x_2) \mapsto (x_2, x_1).$$

By Theorem 3.3,

$$\begin{aligned}\langle M_\alpha, \varphi\rangle &= \frac{1}{\alpha_1 - \alpha_2}\Big(\int_0^\infty \varphi\big(\alpha - t_1\varepsilon_{1,2}\big)dt_1 - \int_0^\infty \varphi\big(\tau(\alpha) - t_2\varepsilon_{1,2}\big)dt_2\Big). \\ &= \int_0^1 \varphi\big((1-t)\alpha + t\tau(\alpha)\big)dt.\end{aligned}$$

Observe that the support of M_α is the segment $[\alpha, \tau(\alpha)]$.

Example 2, n=3

The elementary solution E_3 is given by

$$\begin{aligned}
\langle E_3, \varphi \rangle &= \int_{(\mathbb{R}_+)^3} \varphi(u\varepsilon_{1,2} + v\varepsilon_{2,3} + w\varepsilon_{1,3}) dudvdw \\
&= \int_{(\mathbb{R}_+)^3} \varphi\big((u+w)\varepsilon_{1,2} + (v+w)\varepsilon_{2,3}\big) dudvdw. \\
&= \int_{\{0 \leq w \leq s, 0 \leq w \leq t\}} \varphi(s\varepsilon_{1,2} + t\varepsilon_{2,3}) dsdtdw \\
&= \int_{(\mathbb{R}+)^2} \inf(s,t) \varphi(s\varepsilon_{1,2} + t\varepsilon_{2,3}) dsdt.
\end{aligned}$$

Hence the suppport of E_3 is the angle defined by the rays generated by $\varepsilon_{1,2}$ and $\varepsilon_{2,3}$, with density, if $x = s\varepsilon_{1,2} + t\varepsilon_{2,3}$, $f(s,t) = \inf(s,t)$.

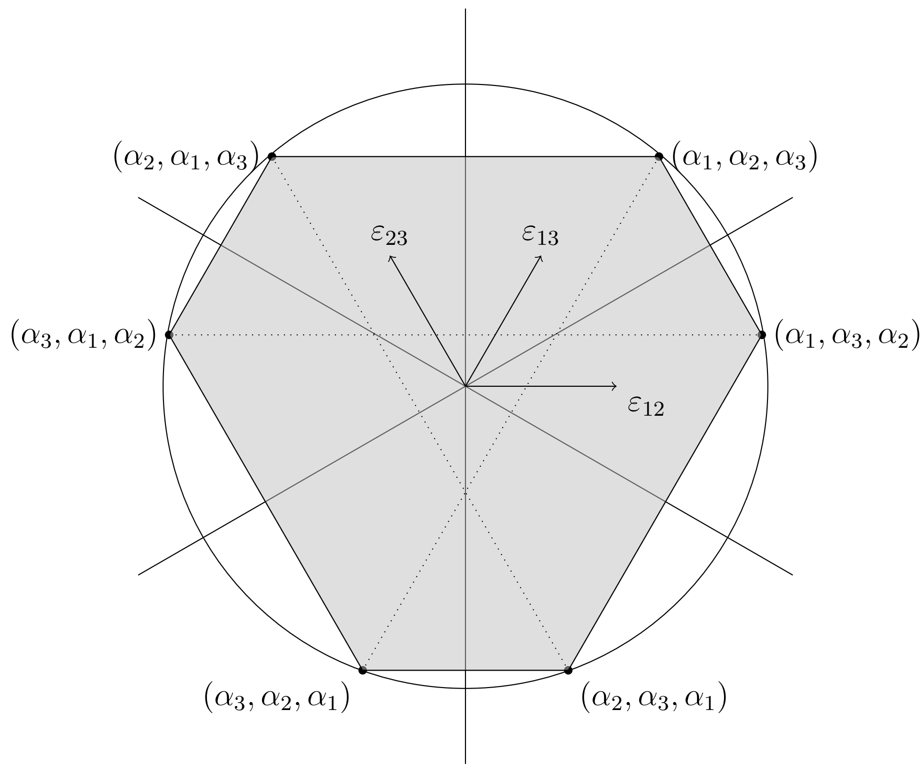


Figure 1. *Heckman's measure, n=3.* For $\alpha_1 > \alpha_2 > \alpha_3$, the support of the measure M_α is the convex hull of the six points $\sigma(\alpha)$ ($\sigma \in \mathfrak{S}_3$). The density of M_α is affine linear in the three trapezia, and in the three lateral triangles, and constant in the middle triangle.

4 The radial part of the convolution product of two orbital measures

Recall that $\nu_{\alpha,\beta}$ denotes the radial part of the convolution product $\mu_\alpha * \mu_\beta$. (The convolution is with respect to $\mathcal{H}_n(\mathbb{F})$.) By Proposition 2.1, the measure $\nu_{\alpha,\beta}$ is determined by the relation

$$\int_{\mathbb{R}^n} \mathcal{E}(z,t)\nu_{\alpha,\beta}(dt) = \mathcal{E}(z,\alpha)\mathcal{E}(z,\beta).$$

Theorem 4.1. *Assume $\mathbb{F} = \mathbb{C}$, the eigenvalues $\alpha_1, \dots, \alpha_n$ to be distinct, and the eigenvalues $\beta_1, \dots, \beta_n$ distinct as well. The radial part $\nu_{\alpha,\beta}$ is given by*

$$\nu_{\alpha,\beta} = \frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\ M_\alpha * \eta_\beta = \frac{1}{n!}\frac{1}{\delta_n!}V_n(x)\ \eta_\alpha * M_\beta,$$

or

$$\nu_{\alpha,\beta} = \frac{1}{n!}\frac{V_n(x)}{V_n(\alpha)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\delta_{\sigma(\alpha)} * M_\beta.$$

The map $(\alpha,\beta) \mapsto \nu_{\alpha,\beta}$ extends continuously as a map $\mathbb{R}^n \times \mathbb{R}^n \to \mathcal{M}_c^1(\mathbb{R}^n)$.

Here the convolutions are with respect to $\mathbb{R}^n$. The measure $\nu_{\alpha,\beta}$ is a $\mathfrak{S}_n$-invariant probability measure on $\mathbb{R}^n$. Observe that

$$\nu_{\alpha,0} = \nu_\alpha = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\delta_{\sigma(\alpha)}.$$

Theorem 4.1 can be seen as a special case of Theorem 2.1 in [Graczyk-Sawyer,2002]. A similar result, but slightly different, is given in [Rösler,2003], p.2436.

Proof. Define $\nu = V_n(x)\ M_\alpha * \eta_\beta$, and let us compute

$$I(z) = \int_{\mathbb{R}^n}\mathcal{E}_n(z,x)\nu(dx).$$

The measure M_α is symmetric, and η_β is skew symmetric, therefore $M = M_\alpha * \eta_\beta$ is skew symmetric, and its Fourier-Laplace transform $\widehat{M}$ as well. We obtain

$$\begin{aligned} I(z) &= \frac{\delta_n!}{V_n(z)}\int_{\mathbb{R}^n}\det\big(e^{z_i x_j}\big)_{1\le i,j\le n}M(dx) \\ &= \frac{\delta_n!}{V_n(z)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\int_{\mathbb{R}^n}e^{(\sigma(z)|x)}M(dx) \\ &= \frac{\delta_n!}{V_n(z)}\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\widehat{M}\big(\sigma(z)\big) = \frac{\delta_n!}{V_n(z)}n!\widehat{M}(z). \end{aligned}$$

Since

$$\widehat{M}(z) = \widehat{M_\alpha}(z)\widehat{\eta_\beta}(z) = \mathcal{E}(z,\alpha)\frac{\delta_n!}{V_n(\beta)}\det\big(e^{z_i\beta_j}\big)_{1\le i,j\le n},$$

we obtain

$$I(z) = n!\delta_n!\mathcal{E}(z,\alpha)\mathcal{E}(z,\beta),$$

which gives the formula of Theorem 4.1. ∎

Recall that $S(\alpha,\beta)$ denotes the support of the measure $\nu_{\alpha,\beta}$. The $\mathfrak{S}_n$-invariant compact set $S(\alpha,\beta) \subset \mathbb{R}^n$ is the set of possible systems of eigenvalues for $C = A + B$, if $\alpha_1, \dots, \alpha_n$ are the eigenvalues of A, and $\beta_1, \dots, \beta_n$ the eigenvalues of B.

Corollary 4.2. (i) *We have the following inclusion :*

$$S(\alpha,\beta) \subset \bigcup_{\sigma\in\mathfrak{S}_n} \big(\sigma(\alpha) + C(\beta)\big).$$

(ii) *If*

$$\min_{i<j}(\alpha_i - \alpha_j) \geq \max_{k,\ell} |\beta_k - \beta_\ell|,$$

then :

$$S(\alpha,\beta) \cap C_n = \alpha + C(\beta).$$

Recall that $C(\beta)$ is the convex hull of the points $\sigma(\beta)$ ($\sigma \in \mathfrak{S}_n$), and C_n is the chamber :

$$C_n = \{t = (t_1, \dots, t_n) \in \mathbb{R}^n \mid t_1 \geq \cdots \geq t_n\}.$$

Part (i) is related to Lidskii's theorem [1950], and can be equivalently written as a system of inequalities

$$\sum_{k\in K} x_k \leq \sum_{i\in I} \alpha_i + \sum_{j\in J} \beta_j,$$

with suitable triples $\{I, J, K\}$. See [Bhatia,2001], p.295, and [Bhatia,1997], Theorem II.1.10.

Proof. a) The support of the measure η_α is the orbit of α under the action of $\mathfrak{S}_n$:

$$\mathrm{supp}(\eta_\alpha) = \{\sigma(\alpha) \mid \sigma \in \mathfrak{S}_n\},$$

and, by Horn's Theorem, the support of Heckman's measure M_β is

$$\mathrm{supp}(M_\beta) = q(\mathcal{O}_\beta) = C(\beta).$$

The statement (i) follows since

$$\mathrm{supp}(\eta_\alpha * M_\beta) \subset \mathrm{supp}(\eta_\alpha) + \mathrm{supp}(M_\beta).$$

In general this is an inclusion and not an equality, because the measure η_α has positive and negative parts, and cancellations are possible.

b) Under the condition

$$\min_{i<j}(\alpha_i - \alpha_j) > \max_{k,\ell} |\beta_k - \beta_\ell|,$$

the sets $\sigma(\alpha) + C(\beta)$ are disjoint, and there is one of them in each chamber $\sigma(C_n)$ ($\sigma \in \mathfrak{S}_n$). Hence no cancellation is possible. □

Theorem 4.1 can be extended as follows. For $\alpha, \beta, \gamma \in \mathbb{R}^n$, the radial part of $\mu_\alpha * \mu_\beta * \mu_\gamma$ is given by

$$\nu_{\alpha,\beta,\gamma} = \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\eta_\alpha * M_\beta * M_\gamma.$$

It generalizes to any finite convolution product. For $\alpha^{(1)}, \ldots, \alpha^{(k)} \in \mathbb{R}^n$, the radial part of $\mu_{\alpha^{(1)}} * \cdots * \mu_{\alpha^{(k)}}$ is given by

$$\nu_{\alpha^{(1)},\ldots,\alpha^{(k)}} = \frac{1}{n!}\frac{1}{\delta_n!} V_n(x)\eta_{\alpha^{(1)}} * M_{\alpha^{(2)}} * \cdots * M_{\alpha^{(k)}}.$$

Example 1, n=2

We use the same notation as in Example 2 of Section 3. We saw that

$$\langle M_\alpha, \varphi\rangle = \int_0^1 \varphi\big((1-t)\alpha + t\tau(\alpha)\big)dt.$$

In this special case, with $a := V_2(\alpha) = \alpha_1 - \alpha_2$, the measure η_α is

$$\eta_\alpha = \frac{1}{a}\big(\delta_\alpha - \delta_{\tau(\alpha)}\big).$$

One can check the following formula for the Fourier-Laplace transform of η_α :

$$\widehat{\eta_\alpha}(z) = e^{\frac{1}{2}(z_1+z_2)(\alpha_1+\alpha_2)}\frac{1}{a}\Big(e^{\frac{a}{2}(z_1-z_2)} - e^{-\frac{a}{2}(z_1-z_2)}\Big).$$

By Theorem 4.1,

$$\nu_{\alpha,\beta} = \frac{1}{2} V_2(x)\ M_\alpha * \eta_\beta = \frac{1}{2} V_2(x)\ \eta_\alpha * M_\beta.$$

Let us explicit the measure $\nu_{\alpha,\beta}$ by using the second expression :

$$\begin{aligned}
&\langle \nu_{\alpha,\beta}, \varphi \rangle \\
&= \frac{1}{2a} \int_0^1 \big(a + (1-2t)b\big) \varphi\big((1-t)(\alpha+\beta) + t(\alpha+\tau(\beta)\big) dt \\
&+ \frac{1}{2a} \int_0^1 \big(a - (1-2t)b\big) \varphi\big((1-t)(\tau(\alpha)+\beta) + t(\tau(\alpha)+\tau(\beta)\big) dt,
\end{aligned}$$

where $b = V_2(\beta) = \beta_1 - \beta_2$. The support $S(\alpha,\beta)$ of $\nu_{\alpha,\beta}$ is the union of two segments. If $a < b$, then

$$S(\alpha,\beta) = [\alpha+\beta, \alpha+\tau(\beta)] \cup [\tau(\alpha)+\beta, \tau(\alpha)+\tau(\beta)].$$

If $a < b$, there are some cancellations, and one obtains

$$S(\alpha,\beta) = [\alpha+\beta, \tau(\alpha)+\beta] \cup [\alpha+\tau(\beta), \tau(\alpha)+\tau(\beta)],$$

and one checks the symmetry $\nu_{\beta,\alpha} = \nu_{\alpha,\beta}$.

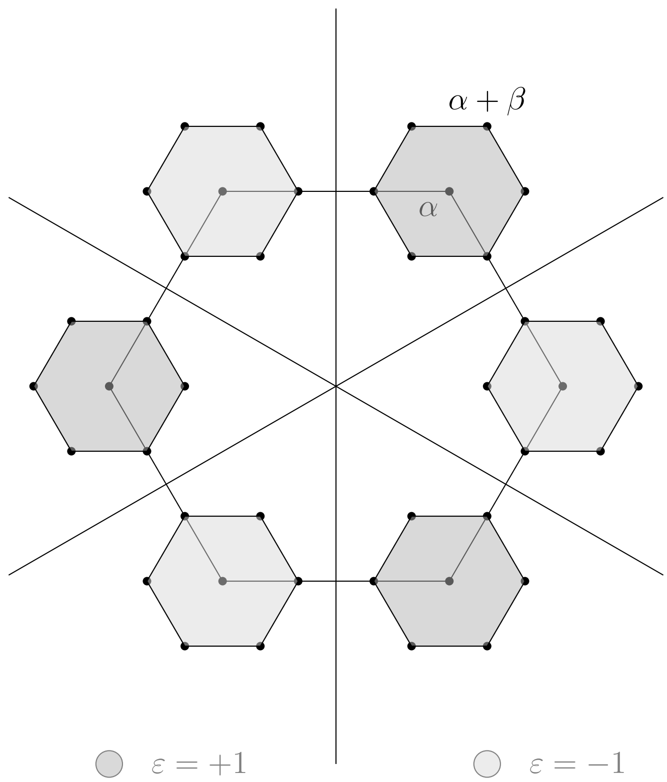


Figure 2. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha = (3, 0, -3)$, $\beta = (1, 0, -1)$. The support is the union of the six hexagons.

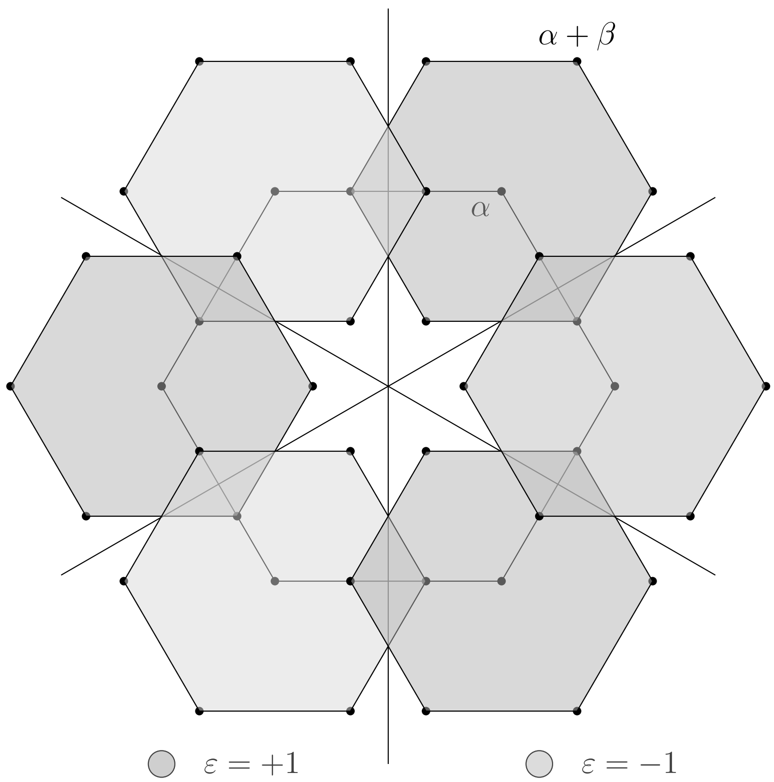
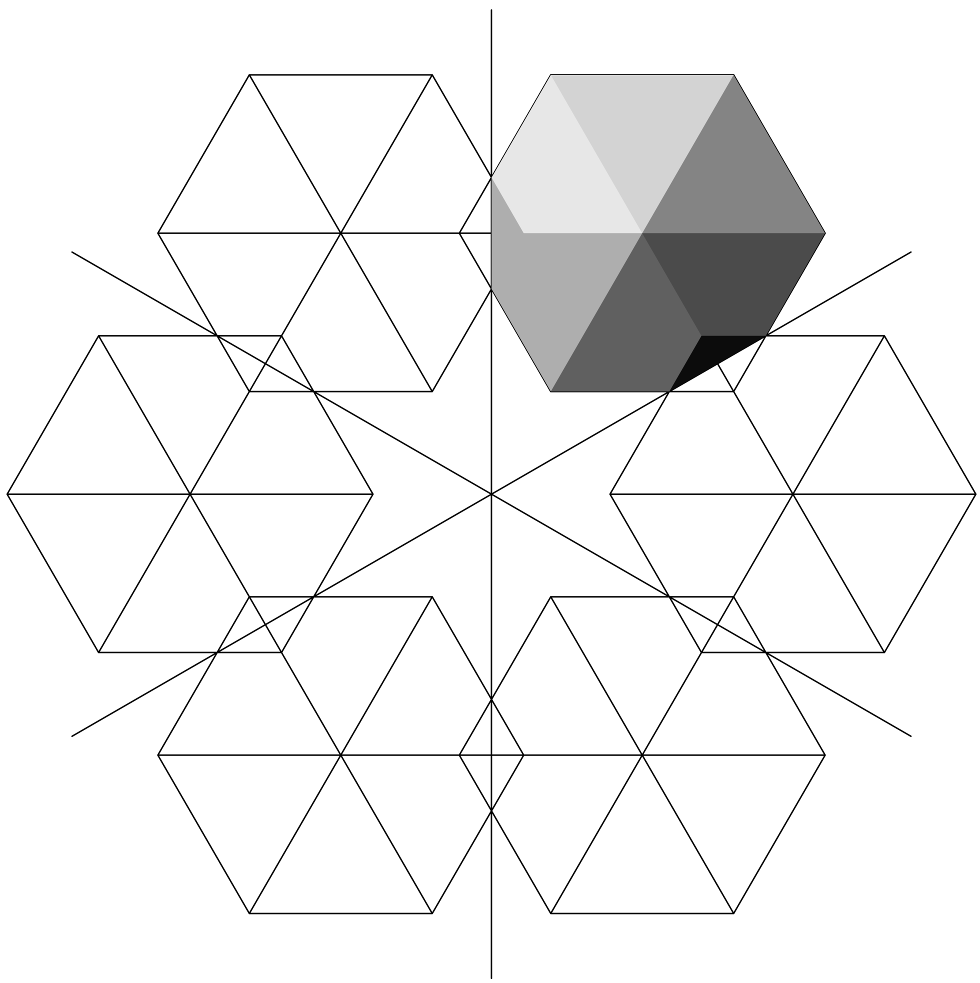


Figure 3. *Support* $S(\alpha,\beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha = (3,0,-3)$, $\beta = (2,0,-2)$. The support is the union of the six hexagons.



5 The case of a rank one matrix B

In this section we consider the special case of a rank one matrix B. In such a case $\beta = (b, 0, \ldots, 0)$ with $b > 0$, or $\beta = (0, \ldots, 0, b)$, with $b < 0$. We assume first that $\beta = (1, 0, \ldots, 0)$. The orbit $\mathcal{O}_\beta$ is the set of Hermitian matrices $Y = (u_i \bar{u}_j)$, vhere $u = (u_1, \ldots, u_n)$ is a unit vector, $u \in S(\mathbb{F}^n)$. In case of $\mathbb{F} = \mathbb{R}$, the orbit $\mathcal{O}_\beta$ can be identified with $S(\mathbb{R}^n)/\{+1, -1\}^n$, and, in case of $\mathbb{F} = \mathbb{C}$, with $S(\mathbb{C}^n)/\mathbb{T}^n$.

Recall that q denotes the projection $q : \mathcal{H}_n(\mathbb{F}) \to D_n \simeq \mathbb{R}^n$. Then

$$q(\mathcal{O}_\beta) = \{(|u_1|^2, \ldots, |u_n|^2) \mid u \in S(\mathbb{F}^n)\}$$

is the simplex $\Sigma_n = \mathrm{Conv}(e_1, \ldots, e_n)$, contained in the hyperplane $x_1 + \cdots + x_n = 1$. The orbital measure μ_β is the image by of the normalized uniform measure on the sphere $S(\mathbb{F}^n)$.

We assume that $\mathbb{F} = \mathbb{C}$ for the rest of this section.

Proposition 5.1. *Heckman's measure $M_\beta = q(\mu_\beta)$ is the normalized uniform measure on the simplex* $\mathrm{Conv}(e_1, \ldots, e_n)$*, i.e. the normalized restriction to the simplex Σ_n of the Lebesgue measure of the hyperplane $x_1 + \cdots + x_n = 1$.*

Proof. The image of the normalized uniform measure on the sphere $S(\mathbb{C}^n)$ under the map

$$S(\mathbb{C}^n) \to \Sigma_n, \quad u \mapsto (|u_1|^2, \ldots, |u_n|^n),$$

is the normalized restriction of the Lebesque measure on the hyperplane $x_1 + \cdots + x_n = 1$ to Σ_n. ∎

Consider on the hyperplane $x_1 + \cdots + x_n = 1$ the differential form

$$w = dx_1 \wedge \cdots \wedge dx_{n-1}.$$

Then

$$\int_{\Sigma_n} w = \frac{1}{(n-1)!},$$

and Heckman's measure M_β can be given by

$$\langle M_\beta, \varphi \rangle = (n-1)! \int_{\Sigma_n} \varphi(x)\, w.$$

Whereas it will not be used in the sequel we give a formula for the Fourier-Laplace transform of Heckman's measure M_β in this special case :

$$\widehat{M_\beta}(z) = \int_{\mathbb{R}^n} e^{(z|x)} M_\beta(dx) = (n-1)! \frac{1}{V_n(z)} \begin{vmatrix} e^{z_1} & \dots & e^{z_n} \\ z_1^{n-2} & \dots & z_n^{n-2} \\ \vdots & & \vdots \\ z_1 & \dots & z_n \\ 1 & \dots & 1 \end{vmatrix}.$$

(This formula can be obtained by using Theorem 4.1 in [Faraut,2015].)

Recall that, for $\alpha = (\alpha_1, \dots, \alpha_n) \in C_n$, $\nu_{\alpha,\beta}$ denotes the radial part of the measure $\mu_\alpha * \mu_\beta$. The following result has been obtained by Frumkin and Goldberger (Theorem 6.1 and Theorem 6.7 in [Frumkin-Goldberger,2006]).

Theorem 5.2. *Assume that* $\beta = (b, 0, \dots, 0)$ *with* $b > 0$.

(i) *The support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$ *is given by*

$$S(\alpha,\beta) \cap C_n =$$
$$\{x \in \mathbb{R}^n \mid x_1 \geq \alpha_1 \geq \dots \geq x_n \geq \alpha_n,\ x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b\}.$$

(ii) *The measure* $\nu_{\alpha,\beta}$ *is absolutely continuous with respect to the Lebesgue measure of the hyperplane* $x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b$ *with the density*

$$h(x) = \frac{1}{n} \frac{1}{b^{n-1}} \frac{1}{V_n(\alpha)}\ V_n(x).$$

(It is assumed that the Lebesgue measure on the hyperplane $x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b$ is associated to the differential form $w = dx_1 \wedge \dots \wedge dx_{n-1}$.)

The inclusion

$$S(\alpha,\beta) \cap C_n$$
$$\subset \{x \in \mathbb{R}^n \mid x_1 \geq \alpha_1 \geq \dots \geq x_n \geq \alpha_n,\ x_1 + \dots + x_n = \alpha_1 + \dots + \alpha_n + b\}.$$

can be found in [Horn-Johnson,1985] (Theorem 4.3.4).

By Theorem 4.1, the density is given in the present case by

$$h(x) = \frac{1}{n} \frac{V_n(x)}{V_n(\alpha)} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \frac{1}{b^{n-1}} \chi\Big(\frac{x - \delta_{\sigma(\alpha)}}{b}\Big),$$

where χ is the indicatrix of the simplex Σ_n.

Let us comment how Theorem 5.2 is related to Theorem 4.1 and Corollary 4.2. The conditions in (i) can be split in two parts :

$$\text{(I)} \quad x_1 \geq \alpha_1, \ldots, x_n \geq \alpha_n, \quad x_1 + \cdots + x_n = \alpha_1 + \cdots + \alpha_n + b.$$

$$\text{(II)} \quad x_2 \leq \alpha_1, \ldots, x_n \leq \alpha_{n-1}.$$

Let us introduce barycentrical coordinates s_i :

$$x_i = \alpha_i + bs_i \quad (i = 1, \ldots, n).$$

Conditions (I) give

$$s_1 \geq 0, \ldots, s_n \geq 0, \; s_1 + \cdots s_n = 1,$$

which mean that $x \in \alpha + b\Sigma_n$. If

$$b \leq \alpha_{i-1} - \alpha_i \quad (i = 2, \ldots, n),$$

then (I) imply (II). Therefore, in this case, $S(\alpha, \beta) \cap C_n = \alpha + b\Sigma_n$.

Observe that the measure $\nu_{\alpha,\beta}$ does not change essentially if one replace $\alpha = (\alpha_1, \ldots, \alpha_n)$ by $(\alpha_1 + c, \ldots, \alpha_n + c)$, and $\beta = (\beta_1, \ldots, \beta_n)$ by $(\beta_1 + d, \ldots, \beta_n + d)$ $(c, d \in \mathbb{R})$. We will write $(\alpha_1 + c, \ldots, \alpha_n + c) \sim (\alpha_1, \ldots, \alpha_n)$. Hence in this section we have considered the case for B to have an eigenvalue of multiplicity $n - 1$, rather than to be of rank one.

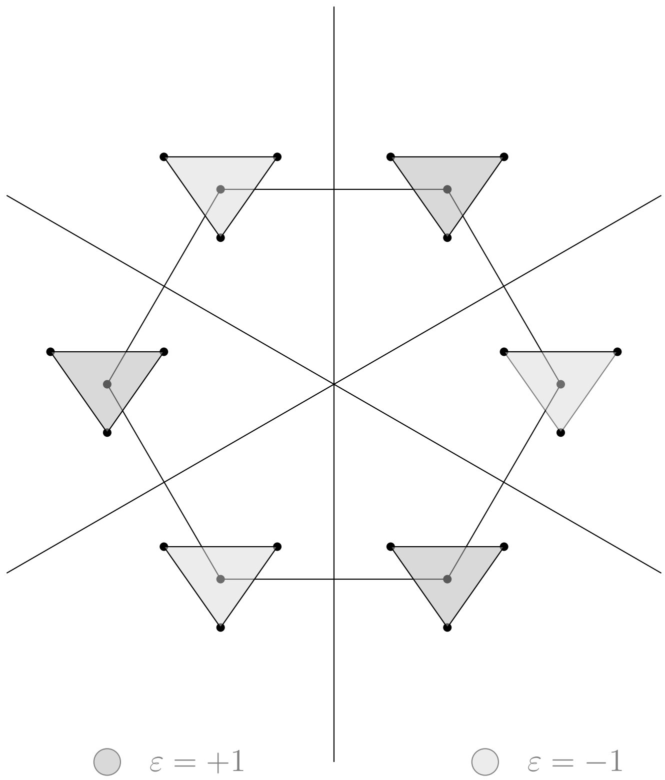


Figure 4 $n = 3$. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$*,* $\alpha = (3, 0, -3)$*,* $\beta = (3, 0, 0) \sim (2, -1, -1)$. The support is the union of the six triangles.

In general there are cancellations which should correspond to conditions (II).

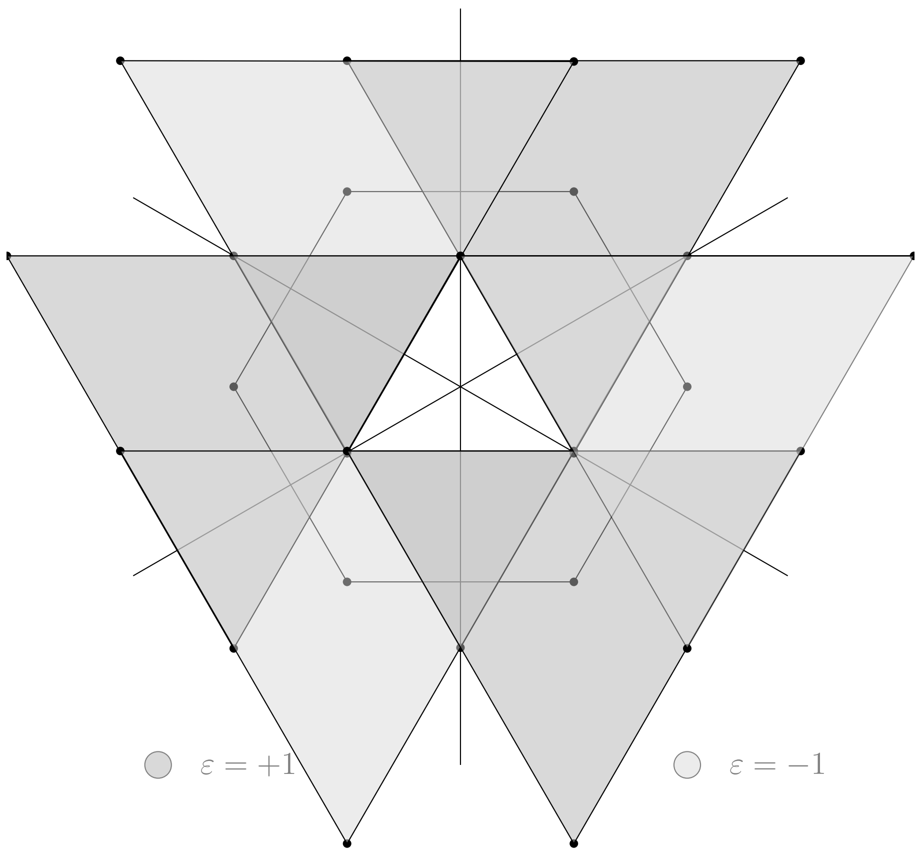


Figure 5 $n = 3$. *Support* $S(\alpha, \beta)$ *of* $\nu_{\alpha,\beta}$, $\alpha = (3, 0, -3)$, $\beta = (6, 0, 0) \sim (4, -2, -2)$. The support is the union of the six triangles, minus the six intersections of two triangles.

6 Real symmetric matrices, $n = 2$

In the case of real symmetric matrices, we know explicitely Heckman's measure and the measure $\nu_{\alpha,\beta}$ only in case of $n = 2$. For $\alpha = (\alpha_1, \alpha_2)$, the orbit $\mathcal{O}_\alpha$ is the set of the matrices

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 \cos^2\theta + \alpha_2 \sin^2\theta & (\alpha_1 - \alpha_2)\cos\theta\sin\theta \\ (\alpha_1 - \alpha_2)\cos\theta\sin\theta & \alpha_1 \sin^2\theta + \alpha_2\cos^2\theta \end{pmatrix}.$$

As in the case of 2×2 Hermitian matrices, the image ot the orbit $\mathcal{O}_\alpha$ under the projection $q : \mathcal{H}_2(\mathbb{R}) \to D_2 \simeq \mathbb{R}^2$ is the segment $[\alpha, \tau(\alpha]$. The projection M_α of the orbital measure μ_α is given by

$$\begin{aligned} \langle M_\alpha, \varphi \rangle &= \frac{1}{2\pi}\int_0^{2\pi} \varphi\big(\alpha_1\cos^2\theta + \alpha_2\sin^2\theta, \alpha_1\sin^2\theta + \alpha_2\cos^2\theta\big) d\theta \\ &= \frac{1}{\pi}\int_0^1 \varphi\big((1-t)\alpha + t\tau(\alpha)\big)\frac{dt}{\sqrt{t(1-t)}}. \end{aligned}$$

Proposition 6.1. *The Fourier-Laplace transform of the orbital measure μ_α is given, if $Z = \mathrm{diag}(z_1, z_2)$, by*

$$\mathcal{F}\mu_\alpha(iZ) = \widehat{M_\alpha}(iz) \int_{\mathbb{R}^2} e^{i(z_1x_1+z_2x_2)} M_\alpha(dx)$$
$$= e^{\frac{i}{2}(z_1+z_2)(\alpha_1+\alpha_2)} J_0\Big(\frac{1}{2}(z_1-z_2)(\alpha_1-\alpha_2)\Big),$$

where J_0 is the Bessel function of index 0.

Proof. By the previous formula

$$\widehat{M_\alpha}(iz) = \frac{1}{\pi}\int_0^1 e^{i\big(z|(1-t)\alpha+t\tau(\alpha)\big)}\frac{dt}{\sqrt{t(1-t)}}.$$

Put $t = \frac{1}{2}(1-\cos\theta)$. Then

$$1 - t = \frac{1}{2}(1+\cos\theta),\ dt = \frac{1}{2}\sin\theta d\theta,$$

and

$$((z \mid (1 - t\alpha + t\tau(\alpha)\big) = \frac{1}{2}(z_1+z_2)(\alpha_1+\alpha_2) + \frac{1}{2}(z_1-z_2)(\alpha_1-\alpha_2)\cos\theta.$$

We obtain

$$\widehat{M_\alpha}(iz) = \frac{1}{\pi}e^{\frac{i}{2}(z_1+z_2)(\alpha_1+\alpha_2)}\int_0^\pi e^{\frac{i}{2}(z_1-z_2)(\alpha_1-\alpha_2)\cos\theta} d\theta.$$

Recall the following integral formula for the Bessel function J_0 :

$$J_0(\zeta) = \frac{1}{\pi}\int_0^\pi e^{i\zeta\cos\theta}d\theta. \qquad \square$$

We introduce the following notation : for $\alpha = (\alpha_1, \alpha_2)$, and $\beta = (\beta_1, \beta_2)$,

$$\tau = \alpha_1 + \alpha_2 + \beta_1 + \beta_2,\ a = \alpha_1 - \alpha_2,\ b = \beta_1 - \beta_2.$$

If a, b, c are the three wedges of a triangle, we denote by $\Delta(a, b, c)$ the area of this triangle. Recall the classical formula :

$$\Delta(a, b, c)^2 = p(p - a)(p - b)(p - c),$$

where p is half the perimeter of the triangle.

Theorem 6.2. *The measure $\nu_{\alpha,\beta}$ is given by*

$$\begin{aligned}&\langle \nu_{\alpha,\beta}, \varphi\rangle = \\ &\frac{1}{2\pi}\int_{|a-b|}^{a+b}\varphi\big(\frac{1}{2}(\tau + r)e_1 + \frac{1}{2}(\tau - r)e_2\big)\frac{2rdr}{\Delta(a, b, r)}. \\ &+\frac{1}{2\pi}\int_{|a-b|}^{a+b}\varphi\big(\frac{1}{2}(\tau - r)e_1 + \frac{1}{2}(\tau + r)e_2\big)\frac{2rdr}{\Delta(a, b, r)}.\end{aligned}$$

Proof. Recall the product formula for the Bessel function J_0 :

$$J_0(\zeta a)J_0(\zeta b) = \frac{1}{\pi}\int_0^\pi J_0(\zeta\sqrt{a^2 + b^2 + 2ab\cos\theta})d\theta.$$

This can be written

$$J_0(\zeta a)J_0(\zeta b) = \int_{|a-b|}^{a+b} J_0(\zeta r)\frac{2rdr}{\sqrt{(2ab)^2 - (a^2 + b^2 - r^2)^2}}.$$

Since

$$(2ab)^2-(a^2+b^2-r^2)^2 = (a+b+r)(a+b-r)(r+a-b)(r-a+b) = 16\Delta(a, b, r)^2,$$

it can also be written

$$J_0(\zeta a)J_0(\zeta b) = \frac{1}{2\pi}\int_{|a-b|}^{a+b} J_0(\zeta r)\frac{rdr}{\Delta(a, b, r)}.$$

It follows that the function $\mathcal{E}(z,\alpha)$ satisfies the following product formula

$$\mathcal{E}(z,\alpha)\mathcal{E}(z,\beta) = \frac{1}{2\pi}\int_{|a-b|}^{a+b} \mathcal{E}(z,\rho)\frac{rdr}{\Delta(a,b,r)},$$

with $\rho = (\rho_1,\rho_2)$, $r = \rho_1 - \rho_2$. By Proposition 2.1, this establishes Theorem 6.2. $\square$

Remarks

In the case of the space of real symmetric matrices $\mathcal{H}_n(\mathbb{R})$, with the action of the orthogonal group $O(n)$, for $n \geq 3$, we don't know any explicit formula for Heckman's measure, and for the measures $\nu_{\alpha,\beta}$. This setting is natural, however the problem is more difficult that in the case of the space of Hermitian matrices, and one should not expect any explicit formula. However the supports should be the same as in the case of $\mathcal{H}_n(\mathbb{C})$ with the action of the unitary group $U(n)$, according to [Fulton,1998], p.265, and [Fulton,2000], Section 10.7.

There should be an analogue of the results presented in this paper in case of pseudo-Hermitian matrices. In this setting, an analogue of Horn's conjecture has been established in [Foth,2010]. An analogue of Theorem 4.1 could probably be obtained by using a formula for the Laplace transform of an orbital measure for the action of the pseudo-unitary group $U(p,q)$ on the space $\mathcal{H}_n(\mathbb{C}^n)$ $(n = p+q)$. This formula is due Ben Saïd and Ørsted [2005]. A related problem has been studied by using this formula in [Faraut,2017].

More generally one could consider Horn's problem for the adjoint action of a compact Lie group on its Lie algebra. The Fourier transform of an orbital measure is explicitely given by the Harish-Chandra integral formula [1957]. Heckman's paper [1982] is written in this framework. One can expect that there is an analogue of Theorem 4.1 in this setting. In particular one can consider the action of the orthogonal group on the space of real skew-symmetric matrices. See [Zuber,2017], and, for a different problem, [Zubov,2016].

One observes some similarity between the results by Frumken and Goldberger [2006], stated in Theorem 5.2 and the classical Cauchy interlacing properties together with Baryshnikov's formula. See [Baryshnikov,2001], and also [Olshanski,2013], [Faraut,2015]. There should be an explanation.

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